SECULAR PERTURBATIONS

ARISING FROM THE

ACTION OF JUPITER ON MARS

A THESIS

Presented to the Faculty of Philosophy of the University of Pennsylvania

BY

ARTHUR BERTRAM TURNER

In Partial Fulfilment of the Requirements for the Degree Doctor of Philosophy

PHILADELPHIA
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CONTENTS.

I.

Lagrange's Generalized Equations of Motion.
Lagrange's Canonical Equations.

Let $F_{11}, F_{12}, F_{13}, \cdots, F_{1n}$ be the forces acting on a unit of mass m_1,

$F_{21}, F_2F_2, F_2F_3, \cdots, F_{2n}$ be the forces acting on a unit of mass m_2,

$$\begin{array}{lr} \vdots & \vdots \\ \text{etc.} & \text{etc.} \end{array}$$

Let $\delta p_{11}, \delta p_{12}, \delta p_{13}, \cdots, \delta p_{1n}$ be the virtual velocities of m_1,

$\delta p_{21}, \delta p_{22}, \delta p_{23}, \cdots, \delta p_{2n}$ be the virtual velocities of m_2,

$$\begin{array}{lr} \vdots & \vdots \\ \text{etc.} & \text{etc.} \end{array}$$

Now assume that each mass m_i be displaced an infinitesimal distance $l = ds_i$ in the direction in which the mass m_i would have moved during the next instant had it not been subjected to this arbitrary displacement, and let the distance in each case be precisely equal to the distance which the body would have moved during the next instant had it not been subjected to displacement. Then by the theorem in virtual velocities that $\sum F\delta p = \delta t =$ change in the living force, we shall have for the masses $m_1 \cdots m_k$,

$$k\sum_1^n m_1 F_{1k}\delta p_{1k} = \delta T \quad \text{for} \quad m_1,$$
$$\vdots \qquad\qquad \vdots$$
$$k\sum_1^n m_k F_{ik}\delta p_{ik} = \delta T \quad \text{for} \quad m_k,$$

adding we get

$$(a) \qquad \delta T = i\sum_1^k k\sum_1^n m_i F_{ik}\delta p_{ik}.$$

These equations involve the masses because F_{ik} are forces on unit mass.

Now it is known that the change in the living force of a system is equal to the work done on the system and since work equals force $\times$ distance, we shall get for the change in the living force

$$(b) \qquad \delta T = i\sum_{1}^{k} m_i \frac{d^2 s_i}{dt^2} \delta s_i$$

Equating these two values of δT, we get,

$$(1) \qquad i\sum_{1}^{k} \left\{ k\sum_{1}^{n} m_i F_{ik} \delta p_{ik} - m_i \frac{d^2 s_i}{dt^2} \delta s_i \right\} = 0$$

which is Lagrange's Generalized Equation.

If now we suppose the forces to be resolved along the three coördinate axes the above equation can be easily made to assume the form,

$$(2) \qquad \sum\left(X - m\frac{d^2 x}{dt^2}\right)\delta x + \sum\left(Y - m\frac{d^2 y}{dt^2}\right)\delta y + \sum\left(Z - m\frac{d^2 z}{dt^2}\right)\delta z = 0$$

where X, Y, Z are the total components of the forces along the coördinate axes.

Let us assume a certain function U (Potential Function) which is independent of the time t, such that

$$\frac{\partial U}{\partial x} = X, \qquad \frac{\partial U}{\partial y} = Y, \qquad \frac{\partial U}{\partial z} = Z;$$

then by substitution equation (2) becomes

$$\sum\left(\frac{\partial U}{\partial x}\delta x + \cdots \text{etc.}\right) = \sum\left(m\frac{d^2 x}{dt^2}\delta x + \cdots \text{etc.}\right).$$

Now the left hand member of this equation is the total variation of U, or δU.

Since T (Living Force) $= \frac{1}{2} mv^2$, $\delta T = mv\delta v$, but

$$m \frac{d^2 x}{dt^2} \delta x = m \frac{dv}{dt} \delta x,$$

and adding

$$m \frac{d^2 x}{dt^2} \delta x = m \frac{dv}{dt} \delta x + mv\delta v - \delta T$$

now

$$\frac{d}{dt}(mv\delta x) = m \frac{dv}{dt} \delta x + mv \frac{d}{dt}(\delta x) = m \frac{dv}{dt} \delta x + mvdv,$$

for

$$mv \frac{d}{dt}(\delta x) = mv\delta \left(\frac{dx}{dt} \right) = mv\delta v.$$

Hence

$$m \frac{d^2 x}{dt^2} \delta x = \frac{d}{dt}(mv\delta x) - \delta T$$

$$\therefore \sum \left(m \frac{d^2 x}{dt} \delta x + \cdots \text{etc.} \right) = \frac{d}{dt}(mv\delta s) - \delta T,$$

or

(3) $$\delta U = \frac{d}{dt}(mv\delta s) - \delta T.$$

Let us suppose T to be a function of the independent variables $q_1, q_2, \cdots$, etc., then the variation of T is

$$\delta T = \frac{\partial T}{\partial q_1} \delta q_1 + \cdots \text{etc.},$$

$$\delta U = \frac{\partial U}{\partial q_1} \delta q_1 + \cdots \text{etc.},$$

$$\delta s = \frac{\partial s}{\partial q_1} \delta q_1 + \cdots \text{etc.}$$

These values substituted in (3) give the equation

$$\left(\frac{\partial U}{\partial q_1} \delta q_1 + \cdots \text{etc.} \right) = \frac{d}{dt}\left(mv \left[\frac{\partial s}{\partial q_1} \delta q_1 + \cdots \text{etc.} \right] \right) - \left(\frac{\partial T}{\partial q_1} \delta q_1 + \cdots \text{etc.} \right)$$

and since the q's are independent we can equate the like variations and obtain the following partial differential equations:—

$$\begin{array}{ccc} \dfrac{\partial U}{\partial q_1} = \dfrac{d}{dt}\left(m\dot{v}\dfrac{\partial s}{\partial q_1}\right) - \dfrac{\partial T}{\partial q_1} \\ \vdots \quad\quad\quad \vdots \quad\quad \vdots \\ \text{etc.} \quad\quad \text{etc.} \quad \text{etc.} \end{array}$$

which become

$$\begin{array}{ccc} \dfrac{\partial U}{\partial q_1} = \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial q_1'}\right) - \dfrac{\partial T}{\partial q_1} \\ \vdots \quad\quad\quad \vdots \quad\quad \vdots \\ \text{etc.} \quad\quad \text{etc.} \quad \text{etc.} \end{array} \tag{4}$$

Since

$$v = \frac{ds}{dt} = \sum \frac{\partial s}{\partial q} \cdot \frac{dq}{dt} + \frac{\partial s}{\partial t} \quad \text{and} \quad \frac{\partial v}{\partial q_1'} = \frac{\partial s}{\partial q_1}.$$

But $\frac{1}{2}mv^2 = T$, therefore

$$\frac{\partial T}{\partial q_1'} = mv\frac{\partial v}{\partial q_1'} = mv\frac{\partial s}{\partial q_1}.$$

These equations are known as Lagrange's Canonical Forms, and in deriving them we have assumed that all points of the system have been expressed in terms of t, and k independent variables $q_1 \cdots q_k$. Since there are $3n$ coördinates altogether in the system, $(x_1\, y_1\, z_1, \cdots, x_n\, y_n\, z_n)$ this assumes that there are $(3n - k)$ equations of condition.

II.

Canonical Forms of Hamilton.

Let us still regard T as expressed in terms of $q, \cdots, q_k$, $q_1', \cdots, q_k'$, and write

$$p_1 = \frac{\partial T}{\partial q_1'}, \quad p_2 = \frac{\partial T}{\partial q_2'}, \quad \cdots, \quad \text{etc.}$$

T was originally a homogeneous function in regard to

$$\frac{dx_1}{dt}, \quad \frac{dx_2}{dt}, \quad \cdots$$

and since dx, dy, dz, $\cdots$ are connected with q'_1, q'_2, $\cdots$ by linear equations, T regarded as a function of q and q' is homogeneous and of the second degree in q'_1, q'_2, $\cdots$. It, therefore, satisfies Euler's equation, or

$$2T_{(q,q')} = q'_1 \frac{\partial T}{\partial q'_1} + q'_2 \frac{\partial T}{\partial q'_2} + \cdots$$

$$= q'_1 p_1 + q'_2 p_2 + \cdots$$

$$= \sum p_i q'_1.$$

Taking the variation of T

$$2\delta T_{(q,q')} = \sum (p\delta q' + q'\,\delta p)$$

and by direct variation

$$\delta T_{(q,q')} = \sum \left(\frac{\partial T_L}{\partial q} \delta q + \frac{\partial T_L}{\partial q'} \delta q' \right),$$

subtracting

$$\delta T_{(q,q')} = \sum \left(q'\,\delta p - \frac{\partial T_L}{\partial q} \delta q \right),$$

but

$$\delta T_{(p,q)} = \sum \left(\frac{\partial T_H}{\partial p} \delta p + \frac{\partial T_H}{\partial q} \delta q \right).$$

Equating like variations we get

$$\begin{array}{c|c} \dfrac{\partial T_H}{\partial p_1} = q'_1 & \dfrac{\partial T_H}{\partial q} = -\dfrac{\partial T_L}{\partial q} \\ \vdots & \vdots \\ \text{etc.} & \text{etc.} \end{array}$$

where

$$T_H = T_{(p,q)} = \phi(p \cdots, q \ \cdots),$$

$$T_L = T_{(q,q')} = F(q \cdots, q' \cdots).$$

Now let $H = T - U$, where $H =$ constant independent of t, then

$$\frac{\partial H}{\partial q_1} = \frac{\partial T_L}{\partial q_1} - \frac{\partial U}{\partial q_1}$$

and equations (4) give

$$\frac{\partial U}{\partial q_1} = \frac{dp_1}{dt} - \frac{\partial T_L}{\partial q_1},$$

subtracting

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial q_1}.$$

Again

$$\frac{\partial H}{\partial p_1} = \frac{\partial T_H}{\partial p_1} - \frac{\partial U}{\partial p_1}.$$

Now U is supposed not to contain q_1', p_1, or t, hence $\partial U/\partial p_1 = 0$, and we have just shown in equations (4) that

$$\frac{\partial T}{\partial p_1} = q_1' = \frac{dq_1}{dt},$$

hence by substitution

$$\frac{\partial H}{\partial p_1} = \frac{dq_1}{dt}.$$

We thus have the systems of equations—

$$(5) \qquad \left\{ \begin{array}{l|l} \dfrac{dp_1}{dt} = -\dfrac{\partial H}{\partial q_1} & \dfrac{dq_1}{dt} = +\dfrac{\partial H}{\partial p_1} \\ \quad\vdots & \quad\vdots \\ \dfrac{dp_k}{dt} = -\dfrac{\partial H}{\partial q_k} & \dfrac{dq_k}{dt} = +\dfrac{\partial H}{\partial p_k} \end{array} \right. .$$

These equations reduce the system of k differential equations of the second order to $2k$ differential equations of the first order. If we call p_i and q_i conjugate independent variables, Hamilton's reduction may be stated thus: "Hamilton's Canonical Forms arise from finding two series of variables in terms of which the coördinates x, y, z, can be expressed. The total differential of any one variable with regard to the time is equal numerically to the partial derivative of a certain determinate function, H, with regard to the conjugate variable."

III.

Method of Jacobi and its Application to Two Bodies. Canonical Constants.

Let us suppose these $2k$ equations of Hamilton to be integrated, then we will get $2k$ constants of integration $c_1 \cdots c_{2k}$, and let us take the partial of H with respect to c_1, since H will be a function of the c's, whence

$$\frac{\partial H}{\partial c_1} = \frac{\partial H}{\partial q_1} \cdot \frac{dq_1}{dc_1} + \cdots + \frac{\partial H}{\partial p_1} \cdot \frac{dp}{dc_1} + \cdots.$$

and by substituting from (5) this becomes

$$\frac{\partial H}{\partial c_1} = \left(\frac{dq_1}{dt} \cdot \frac{dp_1}{dc_1} + \cdots\right) - \left(\frac{dp_1}{dt} \cdot \frac{dq_1}{dc_1} + \cdots\right)$$

$$= \frac{d}{dc_1}\left(p_1 \frac{dq_1}{dt} + \cdots\right) - \frac{d}{dt}\left(p_1 \frac{dq_1}{dc_1} + \cdots\right).$$

But $2T = \sum p_1 (dq_1)/dt$, hence

$$\frac{\partial H}{\partial c_1} = \frac{d(2T)}{dc_1} - \frac{d}{dt}\left(p_1 \frac{dq_1}{dc_1} + \cdots\right).$$

Now $H = T - U$, and substituting, then,

$$\frac{\partial (T + U)}{\partial c_1} = \frac{d}{dt}\left(p_1 \frac{dq_1}{dc_1} + \cdots\right).$$

Integrating with respect to t, we get

$$\frac{\partial}{\partial c_1} \int_{t_0}^{t_1} (T + U)\, dt = \left[p_1 \frac{dq_1}{dc_1} + \cdots\right]_{t_0}^{t_1}.$$

Assume

$$S = \int_{t_0}^{t_1} (T + U)\, dt,$$

and multiply by dc_1, then

$$\partial S = [p_1 dq_1 + \cdots]_{t_0}^{t_1}$$

and

$$\delta S = [p_1 \delta q_1 + \cdots]_{t_0}^{t_1}.$$

Taking the variations of S directly we get

$$\delta S = \frac{\partial S}{\partial q_1}\,\delta q_1 + \cdots + \frac{\partial S}{\partial q_{10}}\,\delta q_{10} + \cdots,$$

and since the q's are independent

$$(6)\qquad \left\{\begin{array}{c|c} \dfrac{\partial S}{\partial q_1} = p_1 & \dfrac{\partial S}{\partial q_{10}} = -\,p_{10} \\ \vdots & \vdots \\ \text{etc.} & \text{etc.} \end{array}\right.$$

We have

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum \frac{\partial S}{\partial q_i}\cdot\frac{dq_i}{dt},\quad \text{also}\quad \frac{dS}{dt} = T + U,\quad \text{and}\quad \frac{\partial S}{\partial q_i} = p_i,$$

$$\therefore T + U = \frac{\partial S}{\partial t} + \sum p_i q_i' = \frac{\partial S}{\partial t} + 2T,$$

$$\therefore \frac{\partial S}{\partial t} = -(T - U) = -H.$$

We here consider H as a function of the $2k$ constants $q_1 \cdots q_k$, and $p_1 \cdots p_k$, but independent of t.

The equation (Jacobi's)

$$(7)\qquad \frac{\partial S}{\partial t} + H\left(q_1 \cdots q_k, \frac{\partial S}{\partial q_1} \cdots \frac{\partial S}{\partial q_k}\right) = 0,$$

when integrated will give S containing the k constants $q_1 \cdots q_k$, and since the partials of S with respect to these k constants are to be put equal to k constants by (6), we shall have introduced upon integrating this last series of k partial differential equations of the first order, $2k$ constants altogether.

To integrate such a differential equation of the first order, we have need of Euler's Transformation, which is derived as follows :—

Suppose $Z = \phi\,(x_1 x_2 x_3 \cdots)$ and we desire to integrate

$$F\left(x_1 x_2 \cdots x_k, \frac{\partial z}{\partial x_1}\cdot\frac{\partial z}{\partial x_2}\cdots\frac{\partial z}{\partial x_k}\right) = 0.$$

Assume $y = z - x_1 \partial z / \partial x_1 = z - x_1 x_1'$, then

$$dy = dz - x_1 dx_1' - x_1' dx_1,$$

but from the equations in z, we have

$$dz = \left(\frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + \cdots \right),$$

and by substitution

$$dy = \left(\frac{\partial z}{\partial x_1} dx_1 + \cdots \right) - x_1 dx_1' - x_1' dx_1,$$

$$= \left(\frac{dz}{dx_2} dx_2 + \cdots \right) - x_1 dx_1'.$$

Therefore

$$\frac{\partial y}{\partial x_1'} = - x_1, \quad \frac{\partial y}{\partial x_2} = \frac{\partial z}{\partial x_2}, \cdots, \text{ etc.},$$

and

$$\frac{\partial y}{\partial x_1} = 0 = \frac{\partial z}{\partial x_1}.$$

These values substituted in the original differential equation, gives

$$F\left(- \frac{\partial y}{\partial x_1'}, x_2, x_3, \cdots x_k, \frac{\partial y}{\partial x_2} \cdots \frac{\partial y}{\partial x_k} \right) = 0.$$

Our new equation contains the same number of variables as the original equation, but the variable x_1 is replaced by $(- \partial y / \partial x_1)$. If this latter is a constant by the conditions of the problem we have thus removed a variable.

It is easily shown that the equations of undistributed motion for two bodies are,

$$\frac{d^2 x}{dt^2} = - \frac{k^2 x}{r^3},$$

$$\frac{d^2 y}{dt^2} = - \frac{k^2 y}{r^3},$$

$$\frac{d^2z}{dt^2} = -\frac{k^2z}{r^3};$$

in which problem

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]$$

$$U = \frac{k^2}{r}$$

$$H = T - U = \tfrac{1}{2}mv^2 - \frac{k^2}{r}.$$

Now Jacobi's equation is

$$\frac{\partial S}{dt} + H\left(q_1 q_2 \cdots, \quad \frac{\partial S}{\partial q_1} \cdots\right) = 0,$$

and hence H must be expressed in terms of the new variables. Let

$$q_1 = x \qquad q_1' = \frac{dx}{dt}$$

$$q_2 = y, \quad \text{then} \quad q_2' = \frac{dy}{dt}, \quad \text{and} \quad T = \tfrac{1}{2}m\,[(q_1')^2 + (q_2')^2 + (q_3')^2]$$

$$q_3 = z \qquad q_3' = \frac{dz}{dt}$$

by substitution

$$\frac{\partial S}{\partial t} + \tfrac{1}{2}\,[(q_1')^2 + (q_2')^2 + (q_3')^2] - \frac{k^2}{r} = 0$$

or

$$\frac{\partial S}{\partial t} + \tfrac{1}{2}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] - \frac{k^2}{r} = 0.$$

Now transform to polar coördinates by means of the equations

$$x = r\cos\sigma\cos\nu$$

$$y = r\cos\sigma\sin\nu$$

$$z = r\sin\sigma \qquad \text{whence}$$

$$\frac{\partial S}{\partial t} + \tfrac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2\cos^2\sigma}\left(\frac{\partial S}{\partial \nu}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \sigma}\right)^2\right] - \frac{k^2}{r} = 0.$$

Now apply Euler's transformation by letting $S' = S + \alpha t$, then $\partial S/\partial t = -\alpha$, [$\alpha$ is a constant of integration] and substituting

$$-\alpha + \tfrac{1}{2}\left[\left(\frac{\partial S'}{\partial r}\right)^2 + \frac{1}{r^2\cos^2\sigma}\left(\frac{\partial S'}{\partial \nu}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S'}{\partial \sigma}\right)^2\right] - \frac{k^2}{r} = 0.$$

Solving we get

$$\frac{\partial S'}{\partial \nu} - \left[2\alpha + \frac{2k^2}{r} - \left(\frac{\partial S'}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S'}{\partial \sigma}\right)^2\right] r^2\cos^2\sigma = 0,$$

again, let $S'' = S' - \alpha_1 \nu$, then $\partial S'/\partial \nu = \alpha_1$ and our equation becomes

$$\alpha_1 - \left[2\alpha + \frac{2k^2}{r} - \left(\frac{\partial S''}{\partial \nu}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S''}{\partial \sigma}\right)\right] r^2\cos^2\sigma = 0$$

which can be written

$$\left[\frac{\alpha_1}{\cos^2\sigma} + \left(\frac{\partial S''}{\partial \sigma}\right)^2\right] = \left[2\alpha r^2 + 2k^2 r - r^2\left(\frac{\partial S''}{\partial r}\right)^2\right]$$

Put the left member $= \alpha_2^2$, then

$$\frac{\alpha_2^2 - 2\alpha r^2 - 2k^2 r}{r^2} = -\left(\frac{\partial S''}{\partial r}\right)^2,$$

or

$$\frac{\partial S''}{\partial r} = \sqrt{2\alpha + \frac{2k^2}{r} - \frac{\alpha_2^2}{r^2}}, \quad \text{and} \quad \left(\frac{\partial S''}{\partial \sigma}\right) = \sqrt{\alpha_2^2 - \frac{\alpha_1}{\cos^2\sigma}}.$$

Hence the complete integral gives

$$S'' = \int_{r_1}^{r} \sqrt{2\alpha + \frac{2k^2}{r} - \frac{\alpha_2^2}{r^2}}\, dr + \int_0^{\sigma} \sqrt{\alpha_2^2 - \frac{\alpha_1}{\cos^2\sigma}}\, d\sigma.$$

Now $S'' = S' - \alpha_1 \nu$, and $S' = S + \alpha t$,

$$\therefore S = \alpha_1 \nu - \alpha t + \int_{r_1}^{r} \sqrt{2\alpha + \frac{2k^2}{r} - \frac{\alpha_2^2}{r^2}}\, dr + \int_0^{\sigma} \sqrt{\alpha_2^2 - \frac{\alpha_1}{\cos^2\sigma}}\, d\alpha.$$

Put

$$\frac{\partial S}{\partial \alpha} = \beta, \quad \frac{\partial S}{\partial \alpha_1} = \beta_1, \quad \text{and} \quad \frac{\partial S}{\partial \alpha_2} = \beta_2,$$

then

$$\beta = -t + \int_{r_1}^{r} \frac{dr}{\sqrt{2\alpha + \frac{2k^2}{r} - \frac{\alpha_2^2}{r^2}}},$$

$$\beta_1 = +\nu - \alpha_1 \int_0^{\sigma} \frac{d\sigma}{\sqrt{\alpha_2^2 - \frac{\alpha_1^2}{\cos^2\sigma}}\cos^2\sigma}$$

$$\beta_2 = -\alpha_2 \int_{r_1}^{r} \frac{dr}{r^2\sqrt{2\alpha + \frac{2k^2}{r} - \frac{\alpha_2^2}{r^2}}} + \alpha_2 \int_0^{\sigma} \frac{d\sigma}{\sqrt{\alpha_2^2 - \frac{\alpha_1^2}{\cos^2\sigma}}},$$

where r_1 is the least root of the equation

$$2\alpha + \frac{2k^2}{r} - \frac{\alpha_2^2}{r^2} = 0.$$

Let $r = r_1$, then $-t = \beta$, or $-\beta =$ Time of Perihelion Passage,

$$\left.\begin{array}{l} \text{If } r_1 = \text{smallest root} \\ \quad r_2 = \text{largest} \quad \text{“} \end{array}\right\} \text{ then } \quad \begin{array}{l} r_1 + r_2 = -\dfrac{k^2}{\alpha} \\ r_1 r_2 = -\dfrac{\alpha_2^2}{2\alpha} \end{array}$$

and

$$r_2 = a(1+e), \quad r_1 + r_2 = 2a,$$
$$r_1 = a(1-e), \quad r_1 r_2 = a^2(1-e^2),$$

hence equating we get

$$2a = -\frac{k^2}{\alpha},$$

$$\therefore \alpha = -\frac{k^2}{2a}, \quad \text{and} \quad \alpha_2 = k\sqrt{a}\sqrt{1-e^2}.$$

From the value of β_1 we see that $\alpha_1^2/\cos^2\alpha = \alpha_2^2$ determines the maximum value of σ (*i. e.*, $\sigma = i$) whence

$$\alpha_1 = \alpha_2 \cos\sigma = k\sqrt{a}\sqrt{1-e^2}\cos i,$$

and when

$$\sigma = 0, \quad \nu = \Omega, \quad \therefore \beta_1 = \Omega.$$

Let APC be a right spherical Δ in which $\angle PAC = i$, $AP = \eta$, and $PC = \sigma$, then $\sin\sigma = \sin i \sin\eta$, and $\cos\sigma d\sigma = \sin i \cos\eta d\eta$, which substituted gives

$$\alpha_2 \int_0^\sigma \frac{d\sigma}{\sqrt{\alpha_2^2 - \dfrac{\alpha_1^2}{\cos^2\sigma}}} = \alpha_2 \int_0^\sigma \frac{d\sigma}{\sqrt{\alpha_2^2\left(1 - \dfrac{\cos^2 i}{\cos^2\sigma}\right)}}$$

$$= \int_0^\sigma \frac{\cos\sigma d\sigma}{\sqrt{\cos^2\sigma - \sin^2 i}} = \int_0^\eta d\eta = \eta.$$

If the body is at perihelion, then $\eta = \omega$, and $\beta_2 = \omega$, or $\beta_2 = \pi - \Omega$. Therefore our six constants of integration or *canonical constants* have the values

$$(8) \qquad \begin{cases} \alpha = -\dfrac{k^2}{2a}, & \beta = -\tau, \\ \alpha_1 = k\sqrt{a}\sqrt{1-e^2}\cos i, & \beta_1 = \Omega, \\ \alpha_2 = k\sqrt{a}\sqrt{1-e^2}, & \beta_2 = \pi - \Omega, \end{cases}$$

IV.

Variation of the Canonical Constants and Jacobi's Equation.

The equations of motion for two bodies are of the form

$$\frac{d^2x}{dt^2} = \frac{\partial U}{\partial x}, \quad \text{where} \quad U = \frac{u}{r}.$$

When a third body is added to the system U is of the form

$$U = \frac{u}{r} + R,$$

where R is the Perturbing Function. The question, therefore, is

to find what change must be made in the canonical constants in order to replace H by $(H-R)$ in the equation just solved.

From Hamilton's Forms, eqts. (5), we have

$$\frac{dq_1}{dt}=\frac{\partial H}{\partial p_1},$$

and if $H \equiv H-R$, becomes

$$\frac{dq_1}{dt}=\frac{\partial(H-R)}{\partial p_1}=\frac{\partial H}{\partial p_1}-\frac{\partial R}{\partial p_1}$$

(9) likewise

$$\frac{dp_1}{dt}=-\frac{\partial H}{\partial q_1}=-\frac{\partial(H-R)}{\partial q_1}=-\frac{\partial H}{\partial q_1}+\frac{\partial R}{\partial q_1}.$$

Considering p_1 and q_1 as functions of the constants and t, we regard the constants as variables and find what variations must take place in them so that H may be replaced by $(H-R)$, that is, p_1 and q_1 must satisfy (9).

Assume

$$q_1=f_1(\alpha\cdots\alpha_k,\ \beta\cdots\beta_k,\ t)$$

$$p_1=F_1(\alpha\cdots\alpha_k,\ \beta\cdots\beta_k,\ t) \qquad \text{then}$$

$$\frac{dq_1}{dt}=\frac{df_1}{dt}+\sum\left(\frac{\partial q_1}{\partial\alpha}\cdot\frac{d\alpha}{dt}+\frac{\partial q_1}{\partial\beta}\cdot\frac{d\beta}{dt}\right)$$

$$\frac{dp_1}{dt}=\frac{dF_1}{dt}+\sum\left(\frac{\partial p_1}{\partial\alpha}\cdot\frac{d\alpha}{dt}+\frac{\partial p_1}{\partial\beta}\cdot\frac{d\beta}{dt}\right)$$

now equation (5) gives

$$\frac{df_1}{dt}=\frac{\partial H}{\partial p_1} \quad \text{and} \quad \frac{dp_1}{dt}=-\frac{\partial H}{\partial q_1},$$

which substituted in (9) gives

(10)

$$-\frac{\partial R}{\partial p_1}=\sum\left(\frac{\partial q_1}{\partial\alpha}\cdot\frac{d\alpha}{dt}+\frac{\partial q_1}{\partial\beta}\cdot\frac{d\beta}{dt}\right)$$

$$\frac{\partial R}{\partial q_1}=\sum\left(\frac{\partial p_1}{\partial\alpha}\cdot\frac{d\alpha}{dt}+\frac{\partial p_1}{\partial\beta}\cdot\frac{d\beta}{dt}\right).$$

Since we can find p and q exclusively in terms of the α's and β's$_1$ and also α and β in terms of the p's and q's, we can apply Jacobi's Theorem which states that,

$$(a)\quad \frac{\partial p_i}{\partial \alpha_k} = \frac{\partial \beta_k}{\partial q_i} \qquad (c)\quad \frac{\partial q_i}{\partial \beta_k} = \frac{\partial \alpha_k}{\partial p_i}$$

$$(b)\quad \frac{\partial p_i}{\partial \beta_k} = -\frac{\partial \alpha_k}{\partial q_i} \qquad (d)\quad \frac{\partial q_i}{\partial \alpha_k} = -\frac{\partial \beta_k}{\partial p_i}$$

whence

$$\frac{\partial q_1}{\partial \alpha} = -\frac{\partial \beta}{\partial p_1}, \quad \frac{\partial q_1}{\partial \beta} = \frac{\partial \alpha}{\partial p_1}, \quad \frac{\partial p_1}{\partial \alpha} = \frac{\partial \beta}{\partial q_1}, \text{ and } \frac{\partial p_1}{\partial \beta} = -\frac{\partial \alpha}{\partial q_1}$$

and making these substitutions in (10)

$$\frac{\partial R}{\partial p_1} = \sum\left(\frac{\partial \beta}{\partial p_1}\cdot\frac{d\alpha}{dt} - \frac{\partial \alpha}{\partial p_1}\cdot\frac{d\beta}{dt}\right)$$

$$\frac{\partial R}{\partial q_1} = \sum\left(\frac{\partial \beta}{\partial q_1}\cdot\frac{d\alpha}{dt} - \frac{\partial \alpha}{\partial q_1}\cdot\frac{d\beta}{dt}\right).$$

If we express R in terms of the p's and q's then by Calculus of Variation, we have

$$\delta R = \sum\left(\frac{\partial R}{\partial p}\delta p + \frac{\partial R}{\partial q}\delta q\right),$$

and substituting from above the values of $\partial R/\partial p$ and $\partial R/\partial q$, we get

$$\delta R = \sum\sum\left\{\left(\frac{\partial \beta}{\partial p}\cdot\frac{d\alpha}{dt} - \frac{\partial \alpha}{\partial p}\cdot\frac{d\beta}{dt}\right)\delta p + \left(\frac{\partial \beta}{\partial q}\cdot\frac{d\alpha}{dt} - \frac{\partial \alpha}{dq}\cdot\frac{d\beta}{dt}\right)\delta q\right\},$$

which can be written

$$\delta R = \sum\sum\left\{\frac{\partial \beta}{\partial q}\delta q + \frac{\partial \beta}{\partial p}\delta p\right\}\frac{d\alpha}{dt} - \sum\sum\left\{\frac{\partial \alpha}{\partial q}\delta q + \frac{\partial \alpha}{\partial p}\delta p\right\}\frac{d\beta}{dt}.$$

Since α and β can be expressed in terms of the p's and q's, let us assume

$$\beta = f(q_1 \cdots q_k, p_1 \cdots p_k),$$

$$\alpha = f_1(q_1 \cdots q_k, p_1 \cdots p_k). \qquad \text{then}$$

$$\delta\beta = \sum \left(\frac{\partial \beta}{\partial q} \delta q + \frac{\partial \beta}{\partial p} \delta p \right), \quad \text{and} \quad \delta\alpha = \sum \left(\frac{\partial \alpha}{\partial q} \delta q + \frac{\partial \alpha}{\partial p} \delta p \right),$$

and by substitution in δR, we have

$$\delta R = \sum \left(\frac{d\alpha}{dt} \delta\beta - \frac{d\beta}{dt} \delta\alpha \right).$$

If now R is expressed in terms of the α's and β's, then

$$\delta R = \sum \left(\frac{\partial R}{\partial \alpha} \delta\alpha + \frac{\partial R}{\partial \beta} \delta\beta \right),$$

and equating like variations, we have

$$(11) \qquad \left\{ \begin{array}{l|l} \dfrac{d\alpha_1}{dt} = \dfrac{\partial R}{\partial \beta_1} & \dfrac{d\beta_1}{dt} = - \dfrac{\partial R}{\partial \alpha_1} \\ \quad \vdots & \quad \vdots \\ \quad \text{etc.} & \quad \text{etc.} \end{array} \right. .$$

These are Jacobi's equations and they give the total variation of the constants in terms of the partial of the Perturbing Function, when the latter has been expressed as a function of the constants (α, β) and the time (t).

V.

Differentiation of the Equations Containing the Canonical Constants.

Solving equations (8) we find

$$(12) \qquad \left\{ \begin{array}{lll} a = - \dfrac{k^2}{2\alpha}, & & \Omega = \beta_1, \\ & & \pi = \beta_1 + \beta_2, \\ \cos i = \dfrac{\alpha_1}{\alpha_2}, & \text{and let} & \epsilon = \pi - n\tau, \\ e^2 = 1 + \dfrac{2\alpha\alpha_2}{k^4}, & \text{then} & \epsilon = \beta_1 + \beta_2 + \dfrac{\beta}{k^2}(-2\alpha)^{\frac{3}{2}}, \\ & \text{since} & n = \dfrac{k}{a^{\frac{3}{2}}}. \end{array} \right.$$

By differentiation of the first of these equations

$$\frac{da}{dt}=\frac{k^2}{2\alpha^2}\cdot\frac{d\alpha}{dt}=\frac{2a^2}{k^2}\frac{d\alpha}{dt},\quad \text{but from (11)}\quad \frac{d\alpha}{dt}=\frac{\partial R}{\partial\beta},$$

$$(13)\qquad \left\{\begin{aligned}
&\therefore\frac{da}{dt}=\left[\frac{2a^2}{k^2}\right]\frac{\partial R}{\partial\beta},\\
&\text{in like manner we can get}\\
&\quad e\frac{de}{dt}=\frac{a\sqrt{1-e^2}}{k^2}\left[\sqrt{1-e^2}\,\frac{\partial R}{\partial\beta}-\frac{k}{a^{\frac{1}{2}}}\frac{\partial R}{\partial\beta_2}\right],\\
&\sin i\frac{di}{dt}=\frac{1}{k\sqrt{a}\sqrt{1-e^2}}\left[\cos i\frac{\partial R}{\partial\beta_2}-\frac{\partial R}{\partial\beta_1}\right],\\
&\quad\frac{d\Omega}{dt}=-\frac{\partial R}{\partial\alpha_1},\\
&\quad\frac{d\pi}{dt}=-\frac{\partial R}{\partial\alpha_1}-\frac{\partial R}{\partial\alpha_2},\\
&\quad\frac{d\epsilon}{dt}=-\frac{\partial R}{\partial\alpha_1}-\frac{\partial R}{\partial\alpha_2}-\frac{k}{a^{\frac{1}{2}}}\frac{\partial R}{\partial\alpha}-\frac{3a}{k^2}(\epsilon-\pi)\frac{\partial R}{\partial\beta}.
\end{aligned}\right.$$

Since ϵ is the only equation in (12) containing β, and R being a function of the α's and β's, then

$$\frac{\partial R}{\partial\beta}=\frac{\partial R}{\partial\epsilon}\cdot\frac{\partial\epsilon}{\partial\beta}=\frac{k}{a^{\frac{1}{2}}}\cdot\frac{\partial R}{\partial\epsilon}.$$

Likewise

$$\frac{\partial R}{\partial\beta_1}=\frac{\partial R}{\partial\epsilon}\cdot\frac{\partial\epsilon}{\partial\beta_1}+\frac{\partial R}{\partial\pi}\cdot\frac{\partial\pi}{\partial\beta_1}+\frac{\partial R}{\partial\Omega}\cdot\frac{\partial\Omega}{\partial\beta_1},$$

but

$$\frac{\partial\epsilon}{\partial\beta_1}=1,\quad\frac{\partial\pi}{\partial\beta_1}=1,\quad\frac{\partial\Omega}{\partial\beta_1}=1.$$

$$\therefore\frac{\partial R}{\partial\beta_1}=\frac{\partial R}{\partial\epsilon}+\frac{\partial R}{\partial\pi}+\frac{\partial R}{\partial\Omega},$$

also

$$\frac{\partial R}{\partial\beta_2}=\frac{\partial R}{\partial\epsilon}\cdot\frac{\partial\epsilon}{\partial\beta_2}+\frac{\partial R}{\partial\pi}\cdot\frac{\partial\pi}{\partial\beta_2},\quad\text{and}\quad\frac{\partial\epsilon}{\partial\beta_2}=1,\quad\frac{\partial\pi}{\partial\beta_2}=1.$$

$$\therefore\frac{\partial R}{\partial\beta_2}=\frac{\partial R}{\partial\epsilon}+\frac{\partial R}{\partial\pi}.$$

In a similar way we get the following as the partials of R with respect to α, α_1, and α_2 respectively:—

$$\frac{\partial R}{\partial \alpha} = \frac{2a^2}{k^2} \cdot \frac{\partial R}{\partial a} + \frac{a}{k^2} \frac{(1-e^2)}{e} \frac{\partial R}{\partial e} - \frac{3a}{k^2}(\epsilon - \pi)\frac{\partial R}{\partial \epsilon},$$

$$\frac{\partial R}{\partial \alpha_1} = -\frac{1}{k\sqrt{a}\sqrt{1-e^2}\sin i} \cdot \frac{\partial R}{\partial i},$$

$$\frac{\partial R}{\partial \alpha_2} = -\frac{1}{k\sqrt{a}} \cdot \frac{\sqrt{1-e^2}}{e} \cdot \frac{\partial R}{\partial e} + \frac{1}{k\sqrt{a}} \cdot \frac{\cos i}{\sqrt{1-e^2}\sin i} \frac{\partial R}{\partial i}.$$

Put $n = k/a^{\frac{3}{2}}$, and substitute these values in (13), then

$$(14)\quad \left\{\begin{aligned}
\frac{da}{dt} &= \frac{2}{na} \cdot \frac{\partial R}{\partial \epsilon}, \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i} \cdot \frac{\partial R}{\partial i}, \\
\frac{d\pi}{dt} &= \frac{\tan\frac{i}{2}}{na^2\sqrt{1-e^2}} \cdot \frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e} \cdot \frac{\partial R}{\partial e}, \\
e\frac{de}{dt} &= -\frac{\sqrt{1-e^2}}{na^2} \cdot \frac{\partial R}{\partial \pi} - \sqrt{1-e^2} \cdot \frac{1-\sqrt{1-e^2}}{na^2} \cdot \frac{\partial R}{\partial \epsilon}, \\
\frac{di}{dt} &= \frac{-1}{na^2\sqrt{1-e^2}\sin i} \cdot \frac{\partial R}{\partial \pi} \\
&\qquad - \frac{\tan\frac{i}{2}}{na^2\sqrt{1-e^2}}\left[\frac{\partial R}{\partial \pi} + \frac{\partial R}{\partial \epsilon}\right]. \\
\frac{\partial \epsilon}{\partial t} &= \frac{-2}{na} \cdot \frac{\partial R}{\partial a} - \frac{\tan\frac{i}{2}}{na^2\sqrt{1-e^2}} \cdot \frac{\partial R}{\partial i} \\
&\qquad + \sqrt{1-e^2}\,\frac{1-\sqrt{1-e^2}}{na^2 e} \cdot \frac{\partial R}{\partial e},
\end{aligned}\right.$$

VI.

Transformation of Equations Expressing the Perturbations and the Values of the Variations.

The perturbations can be expressed in another form by the following substitutions:—

Let the perturbing force which m' exerts upon m at any instant be resolved into three rectangular components as follows:—

(1) $(m'/1+m)R'$ is the component along the radius vector of m_1 reckoned positive away from the sun.

(2) $(m'/1+m)S'$ is the component perpendicular to the radius vector and in the plane of the orbit, positive in the direction of motion.

(3) $(m'/1+m)W'$ is the component perpendicular to the plane of the orbit, positive northward.

Hence each of the variations of R along the coördinate axes will be made up of three parts, and will be determined by the equations,

$$\frac{1+m}{m'}\,\frac{\partial R}{\partial x} = R' \cos R'X + S' \cos S'X + W' \cos W'X,$$

$$\frac{1+m}{m'}\cdot\frac{\partial R}{\partial y} = R' \cos R'Y + S' \cos S'Y + W' \cos W'Y,$$

$$\frac{1+m}{m'}\cdot\frac{\partial R}{\partial z} = R' \cos R'Z + S' \cos S'Z + W' \cos W'Z.$$

The values of the cosines can be obtained from the following spherical triangles:—

Let the plane of XY be the ecliptic (the X axis passing through the vernal equinox) and ♈ the point where it is cut by the plane of the orbit of m; OR', the radius vector of m, then the $\Delta X♈R'$ gives

$$\cos R'X = \cos \Omega \cos u + \sin \Omega \sin u \cos (180 - i),$$

$$= \cos \Omega \cos u - \sin \Omega \sin u \cos i.$$

Let $S'O$ be drawn in the plane of the orbit perpendicular to OR', then $\angle SO\Upsilon = 90 + u$, and $\Delta X\Upsilon S'$ gives

$$\cos S'X = \cos\Omega\cos(90°+u) + \sin\Omega\sin(90°+u)\cos(180 - i),$$

$$= -\cos\Omega\sin u \qquad - \sin\Omega\cos u\cos i.$$

Let $W'O$ be drawn perpendicular to the plane of the orbit, then $\angle W'\Upsilon X = (90 - i)$, and $\Delta X\Upsilon W'$ gives

$$\cos W'X = \cos\Omega\cos 90° + \sin\Omega\sin 90\cos(90 - i),$$

$$= \qquad + \sin\Omega\sin i.$$

On substituting these cosines in our first equation above, we have

$$\frac{1+m}{m'}\frac{\partial R}{\partial x} = R'[\cos u\cos\Omega - \sin u\sin\Omega\cos i]$$

$$+ S'[-\sin u\cos\Omega - \cos u\sin\Omega\cos i] + W'[\sin\Omega\sin i].$$

In like manner we can derive the equations

$$\frac{1+m}{m'}\frac{\partial R}{\partial y} = R'[\cos u\sin\Omega + \sin u\cos\Omega\cos i]$$

$$+ S'[-\sin u\sin\Omega + \cos u\cos\Omega\cos i] + W'[-\cos\Omega\sin i].$$

and

$$\frac{1+m}{m'}\cdot\frac{\partial R}{\partial z} = R'[\sin u\sin i] + S'[\cos u\sin i] + W'[\cos i].$$

By means of the above expressions we can express the partials of the Perturbing Function (R) contained in equations (14) in terms of the components R', S', and W'. As an example we shall find the value of $\partial R/\partial a$:

Now

$$(15)\left\{\frac{\partial R}{\partial a} = \frac{\partial R}{\partial x}\left[\frac{\partial x}{\partial r}\cdot\frac{\partial r}{\partial a}\right] + \frac{\partial R}{\partial y}\left[\frac{\partial y}{\partial r}\cdot\frac{\partial r}{\partial a}\right] + \frac{\partial R}{\partial z}\left[\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial a}\right]\right\}.$$

From the properties of an ellipse, $r = a(1 - e \cos E)$,

$$\therefore \frac{\partial r}{\partial a} = \frac{r}{a}, \quad \text{also} \quad n = \frac{k}{a^{\frac{3}{2}}} = \text{mean daily motion},$$

then

$$E - e \sin E = nt + \epsilon - \pi,$$

$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2},$$

and

$$(16) \qquad \begin{cases} x = r\{[\cos \Omega \cos u] - [\sin \Omega \sin u \cos i]\}, \\ y = r\{[\sin \Omega \cos u] - [\cos \Omega \sin u \cos i]\}, \\ z = r\{[\sin u \sin i]\}, \end{cases}$$

whence we express x, y, and z in terms of the constants α, α_1, α_2, and β, β_1, β_2, and t; likewise, dx/dt, dy/dt, and dz/dt. By differentiating (16) we have

$$\frac{\partial x}{\partial r} = [\cos \Omega \cos u - \sin \Omega \sin u \cos i] = \cos R'X,$$

$$\frac{\partial y}{\partial r} = [\sin \Omega \cos u + \cos \Omega \sin u \cos i] = \cos R'Y,$$

$$\frac{\partial z}{\partial r} = [\sin u \sin i], \qquad = \cos R'Z,$$

and substituting in (15)

$$\frac{\partial R}{\partial a} = \frac{r}{a}\left[\frac{\partial R}{\partial x} \cos R'X + \frac{\partial R}{\partial y} \cos R'Y + \frac{\partial R}{\partial z} \cos R'Z\right].$$

Now we substitute in this equations the value of $\partial R/\partial x$, $\partial R/\partial y$ and $\partial R/\partial z$ derived in the beginning of this article; and it reduces to

$$\frac{\partial R}{\partial a} = \frac{r}{a}\left(\frac{m'}{1+m} R'\right),$$

for the terms containing the squares of the cosines and their products can be reduced by the formulæ

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \quad \cos^2\alpha' + \cos^2\beta' + \cos^2\gamma' = 1,$$

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0,$$

where the cosines are the direction cosines of two lines.

To find $\partial R/\partial i$ we have

$$\frac{\partial R}{\partial i} = \left[\frac{\partial R}{\partial x}\cdot\frac{\partial x}{\partial i}\right] + \left[\frac{\partial R}{\partial y}\cdot\frac{\partial y}{\partial i}\right] + \left[\frac{\partial R}{\partial z}\cdot\frac{\partial z}{\partial i}\right].$$

And from (16) we get

$$\frac{\partial x}{\partial i} = r\left[\sin u \sin\Omega \sin i\right]$$

$$\frac{\partial y}{\partial i} = r\left[-\sin u \cos\Omega \sin i\right]$$

$$\frac{\partial z}{\partial i} = r\left[\sin u \cos i\right]$$

and knowing $\partial R/\partial x$, $\partial R/\partial y$, and $\partial R/\partial z$, $\partial R/\partial i$ reduces to

$$\frac{\partial R}{\partial i} = \frac{m'}{1+m} r W' \sin u.$$

The other partial differential coefficients can be found in a similar manner. If these coefficients be introduced in equations (14), we obtain the following variations of the elements:—

$$
(17)\quad \left\{
\begin{aligned}
\frac{da}{dt} &= \frac{2m'}{1+m}\cdot\frac{na^3}{\sqrt{1-e^2}}\left[\sin vR' + \frac{p}{r}S'\right] \\
\frac{de}{dt} &= \frac{m'}{1+m}\cdot\frac{na^2\sqrt{1-e^2}}{1}\left[\sin vR' \right.\\
&\qquad\qquad \left. + (\cos v + \cos E)\, S'\right] \\
\frac{di}{dt} &= \frac{m'}{1+m}\cdot\frac{na}{\sqrt{1-e^2}}\left[r\cos u\, W'\right] \\
\sin i\,\frac{d\Omega}{dt} &= \frac{m'}{1+m}\cdot\frac{na}{\sqrt{1-e^2}}\left[r\sin u\, W'\right] \\
\frac{d\pi}{dt} &= 2\sin^2\frac{i}{2}\,\frac{d\Omega}{dt} + \frac{d\chi}{dt} \\
e\,\frac{d\chi}{dt} &= \frac{m'}{1+m}\cdot\frac{a^2 n\sqrt{1-e^2}}{1}\left[-\cos vR' \right.\\
&\qquad\qquad \left. + S'(1+r)/p\sin v\right]. \\
\frac{d\epsilon}{dt} &= -2an\frac{m'}{1+m}rR' + \frac{e^2}{1+\sqrt{1-e^2}}\frac{d\pi}{dt} \\
&\qquad\qquad + 2\sqrt{1-e^2}\sin^2\frac{i}{2}\frac{d\Omega}{dt},
\end{aligned}
\right.
$$

or

$$
\frac{d\epsilon}{dt} = -2an\frac{m'}{1+m}rR' + 2\sin^2\frac{\phi}{2}\frac{d\chi}{dt} + 2\sin^2\frac{i}{2}\frac{d\Omega}{dt},
$$

where $e = \sin\phi$.

VII.

Dr. G. W. Hill's[1] First Modification of Gauss's Method.

If the orbits do not intersect each of these differential coefficients may also be obtained in the form of an infinite series arising from the expansion of the Perturbing Function to terms of the first order with respect to the disturbing forces. Since the series contains only terms of the form $A\frac{\sin}{\cos}(iM + i'M')$ in which A is a constant and i and i' positive integers, it follows that the *secular*

[1] On Gauss's Method of Computing Secular Perturbations, by G. W. Hill, Astronomical Paper of American Ephemeris, vol. I.

portion of any differential coefficient will be that corresponding to $i = 0$ and $i' = 0$. If we consider, for example, the coefficient de/dt we will have

$$\frac{de}{dt} = \sum\sum A \tfrac{\sin}{\cos}(iM + i'M') = \sum B \tfrac{\sin}{\cos}(\mu + i'M'),$$

and the part independent of M' will be

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{de}{dt} dM' = \sum C \tfrac{\sin}{\cos}(iM + \nu),$$

the part of this series independent of M is hence

$$\frac{1}{2\pi}\int_0^{2\pi} A_0 dM = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{de}{dt} dM dM' = \left[\frac{de}{dt}\right]_{00}.$$

and this is the secular part of the perturbation.

The computation of the secular part of the perturbations is thus reduced to evaluating the double integrals,

$$\int_0^{2\pi}\int_0^{2\pi} \frac{de}{dt} dM dM'.$$

etc., from the expressions found for them from equations (17) of the last article. The integration with respect of M' can be effected rigorously in terms of elliptic integrals of the first and second species, but that with regard to M can only be approximated to by mechanical quadrature. This quadrature is more accurate if made with regard to E, and we hence transform to this variable by the usual formulæ. The variable M' is replaced by E' also for purposes of symmetry, when we shall have

$$(a) \left[\frac{de}{dt}\right]_{00} = \frac{m'n}{1+m}\frac{\cos\phi}{2\pi}\int_0^{2\pi} [\sin\nu R_0 + (\cos\nu + \cos E)S_0]dE.$$

In which we have written for brevity

$$(b) \quad \left\{ \begin{array}{l} R_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{ar}{m'}(1 - e'\cos E')R'dE', \\ \quad\vdots \\ \quad \text{etc.} \end{array} \right.$$

Gauss's method of effecting the integrations (b) consists in replacing the variable E' by a new variable T which is connected with E' and ten new auxiliaries by the following equations:—

$$\begin{aligned} N \sin E' &= \beta + \beta' \sin T + \beta'' \cos T, \\ N \cos E' &= \alpha + \alpha' \sin T + \alpha'' \cos T, \\ N &= \gamma + \gamma' \sin T + \gamma'' \cos T. \end{aligned}$$

The values α, β, $\gamma \cdots$ are so taken that the coefficients of sin T and cos T vanish in the expression

$$\Delta^2 (\gamma + \gamma' \sin T + \gamma'' \cos T)^2$$

[in which Δ is the distance between the two bodies], which hence takes the form

$$G - G' \sin^2 T + G'' \cos^2 T.$$

This substitution thus finally reduces the integrals of (b) to the form

$$(c) \qquad \int_0^{2\pi} \frac{a \sin^2 T + b \cos^2 T}{(G + G'')^{\frac{3}{2}} [1 - c^2 \sin^2 T)^{\frac{3}{2}}} \, dT,$$

in which a and b are independent of T but involve G, G', and G''. This integral can readily be broken up into elliptic integrals of the first and second kind of which the modulus

$$c^2 = \frac{G' + G''}{G + G''}$$

In the memoir by Dr. G. W. Hill the steps of this reduction will be found given in detail and also very exact tables for effecting the computation. These quantities of the tables are the functions of the elliptic integrals met with in evaluating (c). They are tabulated to the argument $\theta (= \sin^{-1} c)$, and are published to eight decimals, having been computed to ten.

When the values of R_0, S_0 and W_0 have been found, a direct quadrature of (a) will be resorted to to effect the second integration. It is probable that no accuracy is lost by our inability to exactly integrate these expressions since, as is well known, the order of error committed cannot in any of the coefficients exceed a

power of the eccentricities and mutual inclinations of the orbits one has, than the number of parts into which we divide the orbit of the disturbed body.

VIII.

Computation.

The following is an application of Dr. Hill's method to the action of Jupiter upon Mars, and the elements employed are from Dr. Hill's "New Theory of Jupiter and Saturn," pp. 192, 558.

Mars.	Jupiter.
$\pi = 333^\circ\ 17'\ 51''.74$	$\pi' = 11^\circ\ 54'\ 31''.67$
$i = 1\ \ 51\ \ 2.24$	$i' = 1\ \ 18\ \ 42.10$
$\Omega = 48\ \ 23\ \ 54.59$	$\Omega' = 98\ \ 56\ \ 19.79$
$e = 0.09326803$	$e' = 0.04825511$
$n = 689050''.784$	$n' = 109256''.626$
$\log a = 0.1828971$	$\log a' = 0.7162374$
$m = \dfrac{1}{3093500.0}$	$m' = \dfrac{1}{1047.879}$

[Epoch = 1850.0 G.M.T.]

From these elements the preliminary constants become

$I = 1^\circ 26'\ 6''.38$	$\log k = 9.9999971$
$\Pi = 149\ \ 47\ \ 4.37$	$\log k' = 9.9998667$
$\Pi' = 188\ \ 22\ \ 45.43$	$\log c = 8.7995614$
$K = 321\ \ 24\ \ 28.27$	$c = 0.06303204$
$K' = 321\ \ 24\ \ 9.62$	

If the orbit of Mars be divided into twelve parts with regard to the eccentric anomaly, the values of the auxiliary functions corresponding to the several points of division will be as given in the following tables. A rough test of these values is found by comparing the sums of the functions corresponding respectively to the odd and even points of division of the orbit: these are given at the foot of the columns. Sums marked thus (*) indicate that the corresponding numbers have been added instead of their logarithms as given by the points of division.

A test of the perturbations in the plane of the orbit is afforded by the condition that, since

$$\left[\frac{da}{dt}\right]_{00} = 0, \quad \sin\phi \tfrac{1}{2} A_1^s + \cos\phi B_0^s = 0,$$

this residual is found to be + 0.000,000,000,003,2.

The computation has not been duplicated, but various checks on the accuracy of the work have been employed as the computation progressed.

E	Log r	v	A	log B	ϵ
0	0.1403760	0° 0′ 0″.00	29.5201397	0.916245$\bar{4}$	327° 6′ 41″.59
30	0.1463201	32 47 24.62	29.7305725	0.933872$\bar{3}$	355 4 54.78
60	0.1621568	64 44 46.64	29.8340289	0.941863$\bar{6}$	22 20 51.41
90	0.1828971	95 21 5.91	29.8101781	0.9397444	49 30 45.26
120	0.2026919	124 31 47.16	29.6691083	0.9283109	77 5 29.50
150	0.2166314	152 34 23.40	29.4449238	0.9092471	105 33 19.44
180	0.2216237	180 0 0.00	29.1903005	0.885586$\bar{9}$	135 21 49.56
210	0.2166314	207 25 36.60	28.9697704	0.862741$\bar{8}$	166 51 33.40
240	0.2026919	235 28 12.84	28.8461195	0.848552$\bar{0}$	199 55 33.96
270	0.1828971	264 38 54.09	28.8598713	0.8498206	233 38 36.66
300	0.1621568	295 15 13.36	29.0110391	0.8666375	266 35 28.90
330	0.1463201	327 12 35.38	29.2554191	0.8917695	297 46 46.75
S	1.0916971	900 0 0.00	176.0707360	5.3871961	1028 25 54.92
S'	1.0916972	1080 0 0.00	176.0707353	5.3871956	1208 25 56.29

E	Log g	h	l	G	G'
0	0.1016599	27.008366	2.448742	27.0064625	2.4695932
30	8.5335913	27.006191	2.661350	27.0061438	2.6618714
60	9.8433695	27.00766$\bar{8}$	2.76333$\bar{0}$	27.0066000	2.773705$\bar{7}$
90	0.4413039	27.011682	2.735464	27.0074721	2.7765140
120	0.6339503	27.01441$\bar{5}$	2.59166$\bar{2}$	27.0078882	2.6581506
150	0.5856433	27.01325$\bar{5}$	2.36863$\bar{8}$	27.0074658	2.4330414
180	0.2641553	27.009344	2.117924	27.006606$\bar{3}$	2.1522696
210	9.2384099	27.006392	1.900346	27.0061375	1.9039682
240	9.5616845	27.006977	1.776111	27.0064375	1.7842143
270	0.3111659	27.01016$\bar{7}$	1.78667$\bar{3}$	27.0071677	1.8310715
300	0.5312995	27.01257$\bar{5}$	1.93543$\bar{3}$	27.0075528	2.0032709
330	0.4767380	27.01166$\bar{4}$	2.18072$\bar{4}$	27.0071925	2.2348548
S	11.8660187*	162.059344	13.633201	162.0415473	13.841204$\bar{3}$
S'	11.8660198*	162.059350	13.633194	162.0415794	13.8413213

E	G''	θ	Log $\mathfrak{R}$	Log $\mathfrak{T}'$	Log $\mathfrak{R}$
0	0.0189480	17° 39′ 53.″64	0.03165480	0.31498607	0.22325501
30	0.0004748	18 17 56.54	0.03402429	0.31811121	0.22675996
60	0.009308$\bar{8}$	18 43 15.10	0.03565127	0.32025563	0.22916456
90	0.0368400	18 48 58.26	0.03602459	0.32074753	0.22971607
120	0.0599625	18 28 29.55	0.03469754	0.31899872	0.22775521
150	0.0586154	17 39 45.34	0.03164635	0.31497491	0.22324250
180	0.0316073	16 30 39.87	0.02757623	0.30960115	0.21721374
210	0.0033673	15 24 39.12	0.02396320	0.30482483	0.21185327
240	0.0075643	14 55 27.27	0.02245002	0.30282277	0.20960581
270	0.0413999	15 15 16.60	0.02347167	0.30417461	0.21112339
300	0.0628166	16 2 14.88	0.02598814	0.30750244	0.21485859
330	0.0496602	16 53 32.60	0.02889110	0.31133795	0.21916250
S	0.190207$\bar{5}$	102 20 0.31	0.17801800	1.87416678	1.32185292
S'	0.1903576	102 20 8.46	0.17802120	1.87417104	1.32185769

E	Log N	Log P	Log Q	Log V	J_1
0	8.347645$\bar{5}$	5.799087$\bar{0}$	7.139128$\bar{2}$	7.138751$\bar{9}$	27.0207829
30	8.3623563	5.8175271	7.1576461	7.1576367	27.0065656
60	8.395432$\bar{8}$	5.852449$\bar{4}$	7.192977$\bar{9}$	7.192793$\bar{3}$	27.009990$\bar{9}$
90	8.436602$\bar{1}$	5.893197$\bar{8}$	7.234242$\bar{3}$	7.233512$\bar{1}$	27.0290204
120	8.4742977	5.9283888	7.2695991	7.2684105	27.0492441
150	8.499168$\bar{2}$	5.949292$\bar{3}$	7.289985$\bar{3}$	7.288822$\bar{0}$	27.0526533
180	8.5057537	5.951398$\bar{9}$	7.2909894	7.2903609	27.0334422
210	8.4928482	5.9346398	7.2731849	7.2731178	27.0094522
240	8.463346$\bar{7}$	5.902990$\bar{5}$	7.241363$\bar{0}$	7.241212$\bar{3}$	27.0106059
270	8.423946$\bar{6}$	5.863832$\bar{6}$	7.202925$\bar{7}$	7.202101$\bar{4}$	27.0363166
300	8.3844575	5.8269715	7.1668219	7.1655727	27.0521291
330	8.3560126	5.802796$\bar{2}$	7.1428979	7.1419112	27.0423359
S	50.570933$\bar{8}$	35.2612859	43.300879$\bar{4}$	43.297101$\bar{5}$	162.1761951
S'	50.570933$\bar{9}$	35.2612856	43.300882$\bar{1}$	43.297101$\bar{1}$	162.1763440

E	J_2	J_3	Log F_2	Log F_3	R_0
0°	—0.20899495	+0.30968486	0.7664249	9.0593396*n*	0.011523703
30	—0.03630764	—0.02674098	9.9823906	8.3442271*n*	0.011964426
60	+0.15284448	—0.36413193	0.6372797*n*	8.9312244	0.012921849
90	+0.31348410	—0.61207986	0.9362469*n*	8.9588266	0.014185451
120	+0.40064234	—0.70414532	1.0325701*n*	8.3275956*n*	0.015433305
150	+0.38540725	—0.61566127	1.0084166*n*	9.1695220*n*	0.016299428
180	+0.26823871	—0.37034154	0.8476726*n*	9.2218350*n*	0.016504151
210	+0.08088125	—0.03392182	0.3347999*n*	8.7668717*n*	0.015978577
240	—0.12278036	+0.30345688	0.4964372	8.8714541	0.014896494
270	—0.28345025	+0.55139873	0.8711779	9.0346961	0.013593041
300	—0.35632065	+0.64347029	0.9812447	8.2963250	0.012429948
330	—0.32638729	+0.55499846	0.9539640	8.9732022*n*	0.011688529
S	+0.13362957	—0.18200676	—3.6043738*	—0.123045628*	0.083709450
S'	+0.13362742	—0.18200674	—3.6043746*	—0.123045667*	0.083709452

E	S_0	W_0	S_n	R_n
0°	+0.00008004999	+0.00041903914	5.7629857	
30	+0.00001088807	—0.00003989400	4.8906308	7.6305418
60	—0.00007058128	—0.00056153595	5.6865339*n*	7.8866984
90	—0.00013852398	—0.00104078831	5.9586279*n*	7.9689460
120	—0.00017072125	—0.00130819274	6.0295959*n*	7.9232977
150	—0.00015776366	—0.00121033849	5.9813752*n*	7.6945110
180	—0.00010615130	—0.00073761030	5.8043016*n*	
210	—0.00003427562	—0.00006864982	5.3183538*n*	7.6858767*n*
240	+0.00003689362	+0.00053477066	5.3642593	7.9079228*n*
270	+0.00009184620	+0.00088606326	5.7801639	7.9504195*n*
300	+0.00012131643	+0.00094343743	5.9217628	7.8698431*n*
330	+0.00011863616	+0.00076351967	5.9278969	7.6204098*n*
S	—0.00010919379	—0.00071009176	—0.000054778522*	+0.0005847741*
S'	—0.00010919284	—0.00071008769	—0.000054774361*	+0.0005847090*

E	$R_0 \sin v + S_0(\cos v + \cos E)$	$-R_0 \cos v + S_0 \sin v \left(1 + \frac{r}{a \cos^2 \phi}\right)$	Log $W_0 \sin u$.	Log $W_0 \cos u$.
0°	+0.0001601000	—0.0115237021	6.6074024n	6.0323895
30	+0.0064980855	—0.0100466448	5.4290192	5.4698495n
60	+0.0116214735	—0.0056380423	6.0040176	6.7422467n
90	+0.0141365355	+0.0010459980	6.5666029n	6.9896514n
120	+0.0128965787	+0.0084589862	6.9972569n	6.9298457n
150	+0.0077844290	+0.0143154744	7.0724419n	6.4191870n
180	+0.0002123026	+0.0165041521	6.8529748n	6.2779619
210	—0.0072998748	+0.0142155824	5.7054749n	5.6648794
240	—0.0123115656	+0.0083813594	6.2543292	6.7021774n
270	—0.0135423545	+0.0010841018	6.1672230n	6.9414069n
300	—0.0111295720	—0.0055182011	6.7841583n	6.8579918n
330	—0.0061276125	—0.0099498995	6.8612942n	6.3702507n
S	+0.0014493172	+0.0106645521	—0.00243928405*	—0.00233062987*
S'	+0.0014492082	+0.0106646123	—0.00243923736*	—0.00233061181*

E	$-\frac{2r}{a} R_0$
0°	—0.020897817
30	—0.021996061
60	—0.024638500
90	—0.028370902
120	—0.032306045
150	—0.035231951
180	—0.036086917
210	—0.034538413
240	—0.031182353
270	—0.027186081
300	—0.023700582
330	—0.021488837
S	—0.168812214
S'	—0.168812245

If m' is left indefinite, the resulting values of the differential coefficients are:

		Log coeff.
$\left[\frac{de}{dt}\right]_{00}$	$= + \ 165''.71042 m'$	2.2193498
$\left[\frac{d\chi}{dt}\right]_{00}$	$= + 13074.157 \ m'$	4.1164137
$\left[\frac{di}{dt}\right]_{00}$	$= - \ 268.82441 m'$	2.4294687n
$\left[\frac{d\Omega}{dt}\right]_{00}$	$= - \ 8712.3580 \ m'$	3.9401357n
$\left[\frac{d\pi}{dt}\right]_{00}$	$= + 13069.611 \ m'$	4.1162627
$\left[\frac{dL}{dt}\right]_{00}$	$= - 19334.253 \ m'$	4.2863274n

If the above value of m' be employed, we get

$$\left[\frac{de}{dt}\right]_{00} = +\ 0''.15813891, \qquad \left[\frac{d\Omega}{dt}\right]_{00} = -\ 8''.3142788,$$

$$\left[\frac{d\chi}{dt}\right]_{00} = +\ 12.476782, \qquad \left[\frac{d\pi}{dt}\right]_{00} = +\ 12.472445,$$

$$\left[\frac{di}{dt}\right]_{00} = -\ 0.25654148, \qquad \left[\frac{dL}{dt}\right]_{00} = -\ 18.450846.$$

The values of Newcomb are stated on page 378 of his "Secular Variations of the Orbits of the Four Inner Planets." Those of Le Verrier are found in the Annales de l'Observatoire de Paris, Tome II., page 101, and Tome VI., page 189. The results of Le Verrier have been reduced to the above value of m', the three sets of values then compare as follows:

	Results of Le Verrier.	Results of Newcomb.	Method of Gauss.
$\left[\frac{de}{dt}\right]_{00} =$	$+\ 0.15810$	$+\ 0.15818$	$+\ 0.15814,$
$e\left[\frac{d\pi}{dt}\right]_{00} =$	$+\ 1.16323$	$+\ 1.16372$	$+\ 1.16328,$
$\left[\frac{di}{dt}\right]_{00} =$	$-\ 0.25640$	$-\ 0.25655$	$-\ 0.25654,$
$\sin i\left[\frac{d\Omega}{dt}\right]_{00} =$	$-\ 0.26873$	$-\ 0.26850$	$-\ 0.26850,$
$\left[\frac{dL}{dt}\right]_{00} =$	$-\ 18.45027$		$-\ 18.45085.$

BIOGRAPHICAL.

The writer of this thesis was born in Baltimore, Md., August 11, 1872. Prepared at Friends Elementary and High School, Baltimore, Md., he entered the Johns Hopkins University in 1889, from which institution he received the A.B. degree in 1892. Graduate student in Physics, 1892 to February, 1893, when he accepted a position in the U. S. Coast and Geodetic Survey; 1894–1895, Principal of Martin Academy, Kennett Square, Pa.; 1895, to date (including absence on leave 1899–1900), Professor of Mathematics in Temple College, Philadelphia, Pa. While teaching he has pursued graduate work in Astronomy, Mathematics, and Physics at the University of Pennsylvania. Also pursued graduate work in Physics and Mathematics at the Johns Hopkins University, 1899, to February, 1900.